Fixed point of multivalued mapping in uniform spaces

DURAN TÜRKOGLU and BRIAN FISHER*

Department of Mathematics, Faculty of Science and Arts, University of Kirikkale,
71450-Yahsihan Kirikkale, Turkey
*Department of Mathematical Sciences, Faculty of Computer Sciences and
Engineering, De Montfort University, The Gateway, Leicester, LE1 9BH, England
E-mail: duran_t@hotmail.com; fbr@le.ac.uk

Abstract. In this paper we prove some new fixed point theorems for multivalued mappings on orbitally complete uniform spaces.

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1. Introduction

Let $(X, \mathscr{U})$ be a uniform space. A family $\{d_i : i \in I\}$ of pseudometrics on X with indexing set I, is called an associated family for the uniformity $\mathscr{U}$ if the family

$$\beta = \{V(i, \varepsilon) : i \in I; \varepsilon > 0\},$$

where

$$V(i, \varepsilon) = \{(x, y) : x, y \in X, \; d_i(x, y) < \varepsilon\}$$

is a sub-base for the uniformity $\mathscr{U}$. We may assume that β itself is a base by adjoining finite intersection of members of β, if necessary. The corresponding family of pseudo-metrics is called an associated family for $\mathscr{U}$. An associated family for $\mathscr{U}$ will be denoted by p^*. For details the reader is referred to [1,3,4,5,6,7,8].

Let A be a nonempty subset of a uniform space X. Define

$$\Delta^*(A) = \sup \{d_i(x, y) : x, y \in A, i \in I\},$$

where

$$\{d_i : i \in I\} = p^*.$$

Then Δ^* is called an augmented diameter of A. Further, A is said to be p^*-*bounded* if $\Delta^*(A) < \infty$. Let

$$2^X = \{A : A \text{ is a nonempty, closed and } p^* \text{ - bounded subset of } X\}.$$

For any nonempty subsets A and B of X, define

$$d_i(x, A) = \inf \{d_i(x, a) : a \in A\}, i \in I$$

$$H_i(A, B) = \max\left\{\sup_{a \in A} d_i(a, B), \sup_{b \in B} d_i(A, b)\right\}$$
$$= \sup_{x \in X}\left\{|d_i(x, A) - d_i(x, B)|\right\}.$$

It is well-known that on 2^X, H_i is a pseudometric, called the Hausdorff pseudometric induced by d_i, $i \in I$.

Let $(X, \mathscr{U})$ be a uniform space with an augmented associated family p^*. p^* also induces a uniformity $\mathscr{U}^*$ on 2^X defined by the base

$$\beta^* = \{V^*(i, \varepsilon) : i \in I, \varepsilon > 0\},$$

where

$$V^*(i, \varepsilon) = \left\{(A, B) : A, B \in 2^X, H_i(A, B) < \varepsilon\right\}.$$

The space $(2^X, \mathscr{U}^*)$ is a uniform space called the hyperspace of $(X, \mathscr{U})$.

DEFINITION 1.

The collection of all filters on a given set X is denoted by $\Phi(X)$. An order relation is defined on $\Phi(X)$ by the rule $\mathscr{F}_1 < \mathscr{F}_2$ iff $\mathscr{F}_1 \supset \mathscr{F}_2$. If $\mathscr{F}^* < \mathscr{F}$, then $\mathscr{F}^*$ is called a subfilter of $\mathscr{F}$.

DEFINITION 2.

Let $(X, \mathscr{U})$ be a uniform space defined by $\{d_i : i \in I\} = p^*$. If $F : X \to 2^X$ is a multivalued mapping, then

(i) $x \in X$ is called a fixed point of F if $x \in Fx$;

(ii) An orbit of F at a point $x_0 \in X$ is a sequence $\{x_n\}$ given by

$$O(F, x_0) = \{x_n : x_n \in Fx_{n-1}, n = 1, 2, ...\};$$

(iii) A uniform space X is called *F-orbitally complete* if every Cauchy filter which is a subfilter of an orbit of F at each $x \in X$ converges to a point of X.

DEFINITION 3.

Let $(X, \mathscr{U})$ be a uniform space and let $F : X \to X$ be a mapping. A single-valued mapping F is *orbitally continuous* if $\lim(T^{n_i}x) = u$ implies $\lim T(T^{n_i}x) = Tu$ for each $x \in X$.

2. Main results

Theorem 1. *Let $(X, \mathscr{U})$ be an F-orbitally complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$ and $(2^X, \mathscr{U}^*)$ a hyperspace and let $F : X \to 2^X$ be a continuous mapping with Fx compact for each x in X. Assume that*

$$\min\left\{H_i(Fx, Fy)^r, d_i(x, Fx)d_i(y, Fy)^{r-1}, d_i(y, Fy)^r\right\}$$
$$+ a_i \min\{d_i(x, Fy), d_i(y, Fx)\} \leq [b_i d_i(x, Fx)$$
$$+ c_i d_i(x, y)]d_i(y, Fy)^{r-1} \tag{1}$$

for all $i \in I$ and $x, y \in X$, where $r \geq 1$ is an integer, a_i, b_i, c_i are real numbers such that $0 < b_i + c_i < 1$, then F has a fixed point.

Proof. Let x_0 be an arbitrary point in X and consider the sequence $\{x_n\}$ defined by

$$x_1 \in Fx_0, x_2 \in Fx_1, ..., x_n \in Fx_{n-1}, ...$$

Let us suppose that $d_i(x_n, Fx_n) > 0$ for each $i \in I$ and $n = 0, 1, 2, ...$ (Otherwise for some positive integer n, $x_n \in Fx_n$ as desired.)

Let $U \in \mathscr{U}$ be an arbitrary entourage. Since β is a base for $\mathscr{U}$, there exists $V(i, \varepsilon) \in \beta$ such that $V(i, \varepsilon) \subseteq U$. Now $y \to d_i(x_0, y)$ is continuous on the compact set Fx_0 and this implies that there exists $x_1 \in Fx_0$ such that $d_i(x_0, x_1) = d_i(x_0, Fx_0)$. Similarly, Fx_1 is compact so there exists $x_2 \in Fx_1$ such that $d_i(x_1, x_2) = d_i(x_1, Fx_1)$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Fx_n$ and $d_i(x_n, x_{n+1}) = d_i(x_n, Fx_n)$.

For $x = x_{n-1}$, and $y = x_n$ by condition (1) we have

$$\min \left\{ H_i(Fx_{n-1}, Fx_n)^r, d_i(x_{n-1}, Fx_{n-1}) d_i(x_n, Fx_n)^{r-1}, d_i(x_n, Fx_n)^r \right\}$$
$$+ a_i \min \{ d_i(x_{n-1}, Fx_n), d_i(x_n, Fx_{n-1}) \} \le \left[b_i d_i(x_{n-1}, Fx_{n-1}) \right.$$
$$\left. + c_i d_i(x_{n-1}, x_n) \right] d_i(x_n, Fx_n)^{r-1}$$

or since $d_i(x_n, Fx_{n-1}) = 0$, $x_n \in Fx_{n-1}$. Hence we have

$$\min \left\{ d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1} \right\}$$
$$\le \left[b_i d_i(x_{n-1}, x_n) + c_i d_i(x_{n-1}, x_n) \right] d_i(x_n, x_{n+1})^{r-1}$$

and it follows that

$$\min \left\{ d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1} \right\}$$
$$\le (b_i + c_i) d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1}.$$

Since

$$d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1} \le (b_i + c_i) d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1}$$

is not possible (as $0 < b_i + c_i < 1$), we have

$$d_i(x_n, x_{n+1})^r \le (b_i + c_i) d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1}$$

or

$$d_i(x_n, x_{n+1})^r \le k_i d_i(x_{n-1}, x_n) d_i(x_n, x_{n+1})^{r-1},$$

where $k_i = b_i + c_i, 0 < k_i < 1$.

Proceeding in this manner we get

$$d_i(x_n, x_{n+1}) \le k_i d_i(x_{n-1}, x_n)$$
$$\le k_i^2 d_i(x_{n-2}, x_{n-1})$$
$$\vdots$$
$$\le k_i^n d_i(x_0, x_1).$$

Hence we obtain

$$d_i(x_n, x_m) \leq d_i(x_n, x_{n+1}) + d_i(x_{n+1}, x_{n+2}) + \cdots + d_i(x_{m-1}, x_m)$$
$$\leq (k_i^n + k_i^{n+1} + \cdots + k_i^{m-1})\, d_i(x_0, x_1)$$
$$\leq k_i^n (1 + k_i + \cdots + k_i^{m-n-1})\, d_i(x_0, x_1)$$
$$\leq \frac{k_i^n}{1-k}\, d_i(x_0, x_1).$$

Since $\lim\limits_{n \to \infty} k_i^n = 0$, it follows that there exists $N(i, \varepsilon)$ such that $d_i(x_n, x_m) < \varepsilon$ and hence $(x_n, x_m) \in U$ for all $n, m \geq N(i, \varepsilon)$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in the d_i-uniformity on X.

Let $S_p = \{x_n : n \geq p\}$ for all positive integers p and let β be the filter basis $\{S_p : p = 1, 2, ...\}$. Then since $\{x_n\}$ is a d_i-Cauchy sequence for each $i \in I$, it is easy to see that the filter basis β is a Cauchy filter in the uniform space $(X, \mathscr{U})$. To see this we first note that the family $\{V(i, \varepsilon) : i \in I\}$ is a base for $\mathscr{U}$ as $p^* = \{d_i : i \in I\}$. Now since $\{x_n\}$ is a d_i-Cauchy sequence in X, there exists a positive integer p such that $d_i(x_n, x_m) < \varepsilon$ for $m \geq p, n \geq p$. This implies that $S_p \times S_p \subseteq V(i, \varepsilon)$. Thus given any $U \in \mathscr{U}$, we can find an $S_p \in \beta$ such that $S_p \times S_p \subset U$. Hence β is a Cauchy filter in $(X, \mathscr{U})$. Since $(X, \mathscr{U})$ is F-orbitally complete and Hausdorff space, $S_p \to z$ for some $z \in X$. Consequently $F(S_p) \to Fz$ (follows from the continuity of F). Also

$$S_{p+1} \subseteq F(S_p) = \cup\{Fx_n : n \geq p\}$$

for $p = 1, 2, \ldots$. It follows that $z \in Fz$. Hence z is a fixed point of F. This completes the proof.

If we take $r = 1$ in Theorem 1, then we obtain the following theorem.

Theorem 2. *Let $(X, \mathscr{U})$ be an F-orbitally complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$ and $(2^X, \mathscr{U}^*)$ a hyperspace, let $F : X \to 2^X$ be a continuous mapping and Fx compact for each x in X. Assume that*

$$\min\{H_i(Fx, Fy), d_i(x, Fx), d_i(y, Fy)\}$$
$$+ a_i \min\{d_i(x, Fy), d_i(y, Fx)\} \leq b_i d_i(x, Fx) + c_i d_i(x, y) \tag{2}$$

for all $i \in I$ and $x, y \in X$, where a_i, b_i, c_i are real numbers such that $0 < b_i + c_i < 1$, then F has a fixed point.

We denote that if F is a single valued mapping on X, then we can write $d_i(Fx, Fy) = H_i(Fx, Fy), x, y \in X, i \in I$.

Thus we obtain the following theorem as a consequence of the Theorem 2.

Theorem 3. *Let $(X, \mathscr{U})$ be a T-orbitally complete Hausdorff uniform space and let $T : X \to X$ be a T-orbitally continuous mapping satisfying*

$$\min\{d_i(Tx, Ty), d_i(x, Tx), d_i(y, Ty)\}$$
$$+ a_i \min\{d_i(x, Ty), d_i(y, Tx)\} \leq b_i d_i(x, Tx) + c_i d_i(x, y) \tag{3}$$

for all $x, y \in X, i \in I$ and a_i, b_i, c_i are real numbers such that $0 < b_i + c_i < 1$. Then T has a fixed point and which is unique whenever $a_i > c_i > 0$.

Proof. Define a mapping F of X into 2^X by putting $Fx = \{Tx\}$ for all x in X. It follows that F satisfies the conditions of Theorem 2. Hence T has a fixed point.

Now if $a_i > c_i > 0$, we show that T has a unique fixed point. Assume that T has two fixed points z and w which are distinct. Since $d_i(z, Tz) = 0$ and $d_i(w, Tw) = 0$, then by the condition (2),

$$a_i \min \{ d_i(z, Tw), d_i(w, Tz) \} \le c_i d_i(z, w)$$

or

$$a_i d_i(z, w) \le c_i d_i(z, w),$$

$$d_i(z, w) \le \frac{c_i}{a_i} d_i(z, w)$$

which is impossible. Thus if $a_i > c_i > 0$, then T has a unique fixed point in X. This completes the proof.

We note that if $a_i = -1$ in condition (3), then one gets the following result as a corollary.

COROLLARY 4.

Let T be an orbitally cotinuous self-map of a T-orbitally complete uniform space $(X, \mathscr{U})$ satisfying the condition

$$\min \{d_i(Tx, Ty), d_i(x, Tx), d_i(y, Ty)\}$$
$$- \min \{d_i(x, Ty), d_i(y, Tx)\} \le b_i d_i(x, Tx) + c_i d_i(x, y),$$

$x, y \in X$, $i \in I$ and $0 < b_i + c_i < 1$. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Remark 1. If we replace the uniform space $(X, \mathscr{U})$ in Theorem 3 and Corollary 4 by a metric space (i.e. a metrizable uniform space), then Theorem 1 and Corollary 1 of Dhage [2] will follow as special cases of our results.

Acknowledgement

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